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A COMPARISON OF BOUNDS OF DELSARTE AND LOVÁSZ

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# A comparison of bounds of Delsarte and Lovász<sup>\*)</sup>

by

A. Schrijver

## ABSTRACT

We compare two upper bound functions: Delsarte's linear programming bound (an upper bound for the cardinality of cliques in association schemes) and Lovász's  $\theta$ -function (an upper bound for the Shannon capacity of a graph). We show that the two bounds can be treated in a unified fashion. Delsarte's linear programming bound can be generalized to a bound  $\theta'(G)$  for the independence number  $\alpha(G)$  of arbitrary graphs  $G$ , such that  $\theta'(G) \leq \theta(G)$ . On the other hand, if the edge set of  $G$  is the union of some classes of a symmetric association scheme,  $\theta(G)$  may be calculated by means of linear programming. We show that for such graphs  $G$  the product  $\theta(G) \cdot \theta(\bar{G})$  is equal to the number of vertices of  $G$ .

KEY WORDS & PHRASES: *linear programming bound, association scheme, Shannon capacity, positive semi-definite, codes, constant weight codes.*

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## 1. INTRODUCTION

The purpose of this note is to compare two upper bound functions, both being bounds for numbers motivated by more or less information-theoretical problems: Delsarte's linear programming bound, an upper bound for the cardinality of cliques in association schemes, and Lovász's  $\theta$ -function, yielding an upper bound for the Shannon capacity of a graph. The first bound may be conceived of as a bound for the independence number  $\alpha(G)$  of certain graphs  $G$ , whereas Lovász's bound limits  $\alpha(G^k)$ , the independence number of the normal product of  $k$  copies of  $G$ .

We first give, in brief, these two bounds and their theoretical background.

(A *graph* is an undirected graph, without loops or multiple edges.)

Association schemes and Delsarte's linear programming bound (Delsarte [2],

cf. MacWilliams & Sloane [5]). A pair  $(X, \mathcal{R})$ , where  $\mathcal{R} = (R_0, \dots, R_n)$  is a partition of  $X \times X$ , is called a (*symmetric*) *association scheme*, with

*intersection numbers*  $p_{ij}^k$  ( $i, j, k = 0, \dots, n$ ), if

- (1)  $R_0 = \{(x, x) \mid x \in X\}$ ;
- (2)  $R_k^{-1} = \{(y, x) \mid (x, y) \in R_k\} = R_k$ , for  $k = 0, \dots, n$ ;
- (3) for all  $i, j, k = 0, \dots, n$ , and  $(x, y) \in R_k$ :  

$$|\{z \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}| = p_{ij}^k.$$

So  $p_{ij}^k = p_{ji}^k$ . We may consider the pairs  $(X, R_i)$  as graphs ( $i = 1, \dots, n$ ).

$(X, R_i)$  is regular of valency  $v_i = p_{ii}^0$  ( $v_0 = 1$ ). Therefore  $p_{ij}^0 = \delta_{ij} v_i$ .

Let  $D_i$  be the adjacency matrix of  $(X, R_i)$ ;  $D_0$  is the identity matrix. Since,

by (3), the symmetric matrices  $D_0, \dots, D_n$  commute there exists a matrix

$P = (p_{k,u}^u)_{k,u=0}^n$  such that  $p_k^0, \dots, p_k^n$  are the eigenvalues of  $D_k$  ( $k = 0, \dots, n$ ),

and the eigenvalues  $P_0^u, \dots, P_n^u$  of  $D_0, \dots, D_n$ , respectively, have a common eigenvector  $(u=0, \dots, n)$ . We may assume that  $P_k^0 = v_k$  for all  $k$ . Set

$$(4) \quad Q_k^u = \frac{\mu_u}{v_k} P_k^u,$$

where  $\mu_u$  is the dimension of the common eigenspace of  $D_0, \dots, D_n$  belonging to  $P_0^u, \dots, P_n^u$ , respectively  $(u=0, \dots, n)$ . It can be shown that

$$(5) \quad \sum_{u=0}^n P_k^u Q_\ell^u = m \cdot \delta_{k\ell} \quad \text{and} \quad \sum_{k=0}^n P_k^u Q_k^v = m \cdot \delta_{uv},$$

where  $m = |X|$ . So  $P$  and  $m^{-1} \cdot Q^T$  represent inverse matrices.

Coding theorists are interested in two families of association schemes:

the families of Hamming schemes and Johnson schemes, respectively.

Let  $n$  and  $q$  be natural numbers and let  $X$  be the set of vectors of length  $n$ , with entries in  $\{0, \dots, q-1\}$ . Moreover let, for  $k = 0, \dots, n$ :

$$(6) \quad R_k = \{(x, y) \in X \times X \mid d_H(x, y) = k\},$$

where  $d_H(x, y)$  denotes the *Hamming distance* between the vectors  $x$  and  $y$ ,

i.e. the number of coordinate places in which  $x$  and  $y$  differ. Let

$R = (R_0, \dots, R_n)$ . As can be checked easily  $(X, R)$  is a symmetric association scheme; schemes obtained in this way are called *Hamming schemes*.

For Hamming schemes the values of  $v_k$ ,  $\mu_u$  and  $P_k^u$  are given by:

$$(7) \quad v_k = \binom{n}{k} \cdot (q-1)^k, \quad \mu_u = \binom{n}{u} \cdot (q-1)^u,$$

$$P_k^u = K_k(u) = \sum_{j=0}^k (-1)^j (q-1)^{k-j} \binom{u}{j} \binom{n-u}{k-j} = \sum_{j=0}^k (-q)^j (q-1)^{k-j} \binom{n-j}{k-j} \binom{u}{j},$$

for  $k, u = 0, \dots, n$  ( $K_k(u)$  is the *Krawtchouk polynomial* of degree  $k$  in the

variable  $u$ ).

The second family is obtained as follows. Let  $v$  and  $n$  be natural numbers and let  $X$  be the set of 0,1-vectors of length  $v$  with exactly  $n$  ones ( $n \leq \frac{1}{2}v$ ). Moreover, let, for  $k = 0, \dots, n$ ,

$$(8) \quad R_k = \{(x, y) \in X \times X \mid d_J(x, y) = k\},$$

where  $d_J(x, y) = \frac{1}{2}d_H(x, y)$  is the *Johnson distance* between  $x$  and  $y$ . Let  $\mathcal{R} = (R_0, \dots, R_n)$ . Then  $(X, \mathcal{R})$  is a symmetric association scheme; schemes constructed in this manner are called *Johnson schemes*. Their parameters are:

$$(9) \quad v_k = \binom{n}{k} \binom{v-n}{n-k}, \quad \mu_u = \binom{v}{u} - \binom{v}{u-1} = \frac{v-2u+1}{v-u+1} \binom{v}{u},$$

$$P_k^u = E_k(u) = \sum_{j=0}^k (-1)^{k-j} \binom{n-j}{k-j} \binom{n-u}{j} \binom{v-n+j-u}{j} = \sum_{j=0}^k (-1)^j \binom{u}{j} \binom{n-u}{k-j} \binom{v-n-u}{k-j},$$

for  $k, u = 0, \dots, n$  ( $E_k(u)$  is the *Eberlein polynomial* of degree  $2k$  in the variable  $u$ ).

(A third family of symmetric association schemes is formed by *strongly regular graphs*. These are exactly those graphs  $(X, R_1)$  such that  $(X, \mathcal{R})$  is a symmetric association scheme, where  $\mathcal{R} = (R_0, R_1, R_2)$ ,  $R_2 = (X \times X) \setminus (R_0 \cup R_1)$ . It follows that the complementary graph of a strongly regular graph is strongly regular.)

The main problem in combinatorial coding theory is to estimate the maximum size of any subset  $C$  (a "code") of (the set  $X$  in) Hamming and Johnson schemes such that no two elements in  $C$  have (Hamming or Johnson) distance less than a given value  $d$ . A generalized translation of this problem into the language of association schemes needs the notion of an  $M$ -clique; given  $0 \in M \subset \{0, \dots, n\}$  a subset  $Y$  of  $X$  is an  $M$ -clique if  $(x, y) \in \bigcup_{k \in M} R_k$  for all  $x, y \in Y$ .

So the coding problem is to determine the maximum cardinality of  $\{0, d, d+1, \dots, n\}$ -cliques in Hamming and Johnson schemes.

To obtain an upper bound for the size of cliques in a symmetric association scheme  $(X, \mathcal{R})$  define, for  $Y \subset X$ , the *inner distribution*  $(a_0, \dots, a_n)$  of  $Y$  by

$$(10) \quad a_k = \frac{|R_k \cap (Y \times Y)|}{|Y|},$$

for  $k = 0, \dots, n$ ; so  $a_0 = 1$  and  $\sum_{k=0}^n a_k = |Y|$ . Moreover, if  $Y$  is an  $M$ -clique then  $a_k = 0$  if  $k \notin M$ . Delsarte showed that, for the inner distribution of any subset  $Y$  of  $X$ , one has

$$(11) \quad \sum_{k=0}^n a_k Q_k^u \geq 0,$$

for  $u = 0, \dots, n$ . Therefore, for  $M$ -cliques  $Y$  one has

$$(12) \quad |Y| \leq \max \left\{ \sum_{k=0}^n a_k \mid a_0, \dots, a_n \geq 0; a_0 = 1; a_k = 0 \text{ for } k \notin M; \sum_{k=0}^n a_k Q_k^u \geq 0 \right\} = \\ = \min \left\{ \sum_{u=0}^n b_u \mid b_0, \dots, b_n \geq 0; b_0 = 1; \sum_{u=0}^n b_u P_k^u \leq 0 \text{ for } k \in M \setminus \{0\} \right\}.$$

The equality in (12) follows from the duality theorem of linear programming. This bound on the size of cliques is called *Delsarte's linear programming bound*. One may apply linear programming techniques to calculate its value - see [1] for applications in coding theory.

The following result of Delsarte shows that the linear programming bound is a sharpening of the Hamming bound in coding theory. Let  $(X, \mathcal{R})$  be a symmetric association scheme, with  $\mathcal{R} = (R_0, \dots, R_n)$ , and let  $0 \in M \subset \{0, \dots, n\}$  and  $\bar{M} = \{0\} \cup (\{0, \dots, n\} \setminus M)$ . Then

(13) the product of the linear programming bound for  $M$ -cliques and the linear programming bound for  $\overline{M}$ -cliques is at most  $|X|$ .

Hence  $|Y| \cdot |Z| \leq |X|$  for  $M$ -cliques  $Y$  and  $\overline{M}$ -cliques  $Z$ . Taking  $M = \{0, d, d+1, \dots, n\}$  in a Hamming scheme the Hamming bound follows.

The Shannon capacity and Lovász's bound. Lovász [4] introduced, for any graph  $G$ , the number  $\Theta(G)$ , which is an upper bound for the "Shannon capacity"  $\Theta(G)$ . Let  $\alpha(G)$  be the maximum number of independent (i.e. pairwise non-adjacent) points in a graph  $G$ , and let  $G \cdot H$  denote the (normal) product of graphs  $G$  and  $H$ , i.e. the point set of  $G \cdot H$  is the cartesian product of the point sets of  $G$  and  $H$ , whereas two distinct points of  $G \cdot H$  are adjacent iff in both coordinate places the elements are adjacent or equal.  $G^k$  denotes the product of  $k$  copies of  $G$ .

Shannon [9] introduced the following number for graphs  $G$ :

$$(14) \quad \Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k \rightarrow \infty} \sqrt[k]{\alpha(G^k)},$$

which is called the *Shannon capacity* of  $G$ .

If one considers the points of  $G$  as letters in an alphabet, two points being adjacent iff they are "confoundable", then  $\alpha(G^k)$  may be interpreted as the maximum number of  $k$ -letter messages such that any two of them are inconfoundable in at least one coordinate place.

Since  $\alpha(G)^k \leq \alpha(G^k)$ , it follows that  $\alpha(G) \leq \Theta(G)$ . Equality does not hold in general; e.g.  $\alpha(C_5) = 2$ , whereas  $\alpha(C_5^2) = 5 \leq \Theta(C_5)^2$ . Lovász showed that, in fact,  $\Theta(G) = \sqrt{5}$ . Actually, he gave a general upper bound for  $\Theta(G)$  as follows.

Let  $G = (V, E)$  be a graph, with vertex set  $V = \{1, \dots, n\}$ , and define



$$(15) \quad \theta(G) = \min\{\ell\text{eva} \mid A=(a_{ij}) \text{ is a symmetric } n \times n\text{-matrix such that} \\ a_{ij} = 1 \text{ if } \{i,j\} \in E\},$$

where  $\ell\text{eva}$  denotes the largest eigenvalue of  $A$ . Now, if  $\alpha(G) = k$ , each matrix  $A$  satisfying the conditions mentioned in (15) has a  $k \times k$  all-one principal submatrix (with largest eigenvalue  $k$ ), hence  $\ell\text{eva} \geq k$ . Therefore  $\alpha(G) \leq \theta(G)$ . Since, as Lovász proved,  $\theta(G \cdot H) = \theta(G) \cdot \theta(H)$  for all graphs  $G$  and  $H$ , one has  $\alpha(G^k) \leq \theta(G)^k$ , which yields the stronger inequality  $\Theta(G) \leq \theta(G)$  (Haemers [3] showed the existence of graphs  $G$  with  $\Theta(G) < \theta(G)$ ). Moreover Lovász showed

$$(16) \quad \theta(G) = \max\{\sum_{i,j} b_{ij} \mid B=(b_{ij}) \text{ is an } n \times n \text{ positive semi-definite} \\ \text{matrix, with } \text{Tr} B=1, \text{ and } b_{ij}=0 \text{ whenever } \{i,j\} \in E\}.$$

So  $\theta(G)$  may be considered as both a maximum and a minimum, which makes the function  $\theta$  easier to handle. Lovász found, inter alia, for graphs  $G$  (with  $n$  points):

$$(17) \quad \theta(G) \cdot \theta(\overline{G}) \geq n \text{ (where } \overline{G} \text{ is the complementary graph), with equality} \\ \text{if } G \text{ is vertex-transitive;}$$

and

$$(18) \quad \theta(G) \leq \frac{-n\lambda_n}{\lambda_1 - \lambda_n} \text{ if } G \text{ is regular } (\lambda_1 \text{ and } \lambda_n \text{ being the largest and} \\ \text{smallest eigenvalues of the adjacency matrix of } G), \text{ with equality} \\ \text{if } G \text{ is edge-transitive.}$$

A consequence of (18) is: let  $v \geq 2n$  and let  $K(v,n)$  be the graph whose vertices

are the  $n$ -subsets of some fixed  $v$ -set, two vertices being adjacent iff they are disjoint; such graphs are called *Kneser-graphs*. Then

$$(19) \quad \theta(K(v,n)) = \binom{v-1}{n-1}$$

(by (18) it is sufficient to calculate the eigenvalues of  $K(v,n)$ ), generalizing the Erdős-Ko-Rado theorem, which says that  $\alpha(K(v,n)) = \binom{v-1}{n-1}$ .

The theories of Delsarte and Lovász appear to have certain common characteristics, such as bounding cliques or independent sets in graphs, using eigenvalue-techniques on matrices determined by graphs, yielding relations between a graph and its complement, and being applicable to allied structures such as "constant weight codes" and Kneser-graphs. The purpose of this note is to go further into this relationship.

Clearly, Delsarte's linear programming bound may be conceived of as an upper bound for  $\alpha(G)$  for graphs  $G$  whose edge set is the union of some classes  $R_i$  of a symmetric association scheme  $(X, \mathcal{R})$ . We show that Delsarte's bound can be extended to a bound  $\theta'(G)$  for  $\alpha(G)$  for arbitrary graphs  $G$ ; the description of  $\theta'(G)$  has many features in common with Lovász's  $\theta(G)$ . It will follow that  $\theta'(G) \leq \theta(G)$  (in general  $\theta'(G) \neq \theta(G)$ ). On the other hand, if the edge set of  $G$  is the union of some classes of a symmetric association scheme  $(X, \mathcal{R})$  the number  $\theta(G)$  may be calculated by means of a linear program obtained from (12) by dropping the nonnegativity constraints for  $a_0, \dots, a_n$ . It follows that also for such graphs  $G$  one has  $\theta(G) \cdot \theta(\bar{G}) = |X|$  (cf. (13) and (17)).

## 2. A COMPARISON OF THE BOUNDS OF DELSARTE AND LOVÁSZ

First recall the following strong form of the duality theorem of linear programming. Let  $C$  and  $D$  be closed convex cones in  $\mathbb{R}^k$  and  $\mathbb{R}^m$ , respectively, with dual cones  $C^*$  and  $D^*$ , respectively (that is,  $C^*$  consists of all vectors in  $\mathbb{R}^k$  having a nonnegative inner product with each element of  $C$ ). Let  $M$  be a real-valued  $m \times k$ -matrix, and let  $c \in \mathbb{R}^k$  and  $d \in \mathbb{R}^m$ . Then

$$(20) \quad \max\{cx \mid x \in C; d - Mx \in D\} = \min\{yd \mid y \in D^*; yM - c \in C^*\},$$

provided that the object sets are nonempty and closed. Furthermore, notice that the closed convex cone of all real-valued symmetric positive semi-definite  $n \times n$ -matrices, conceived of as  $n^2$ -vectors, has as dual cone the set of real-valued  $n \times n$ -matrices  $U$  such that  $y^T U y \geq 0$  for all real  $n$ -vectors  $y$ . (So symmetric matrices in the dual cone are positive semi-definite.) For convenience, we use the following inner product notation for  $n \times n$ -matrices

$A = (a_{ij})$  and  $B = (b_{ij})$ :

$$(21) \quad A * B = \sum_{i,j=1}^n a_{ij} \cdot b_{ij},$$

that is,  $A * B = \text{Tr}(A^T B)$ . So  $A * I = \text{Tr} A$  and  $A * J = \sum_{i,j=1}^n a_{ij}$ .

Let  $G$  be a graph, with point set  $\{1, \dots, n\}$ . Lovász defined

$$(22) \quad \theta(G) = \max\left\{\sum_{i,j} b_{ij} \mid B = (b_{ij}) \text{ is a symmetric positive semi-definite } n \times n\text{-matrix with } \text{Tr} B = 1, \text{ and } b_{ij} = 0 \text{ if } \{i,j\} \in E\right\} = \\ \min\{\text{Lev} A \mid A = (a_{ij}) \text{ is a symmetric } n \times n\text{-matrix with } a_{ij} = 1 \text{ if } \{i,j\} \notin E\}.$$

Now define  $\theta'(G)$  as follows.

$$(23) \quad \theta'(G) = \max\{\sum_{i,j} b_{ij} \mid B=(b_{ij}) \text{ is a nonnegative symmetric positive semi-definite } n \times n\text{-matrix with } \text{Tr} B=1, \text{ and } b_{ij}=0 \text{ if } \{i,j\} \in E\},$$

so the difference with (22) is the restriction of the range for the maximum to nonnegative matrices  $B$ .

THEOREM 1.  $\alpha(G) \leq \theta'(G) \leq \theta(G)$ .

PROOF. Clearly,  $\theta'(G) \leq \theta(G)$ . Suppose  $Y \subset \{1, \dots, n\}$  is an independent set with  $\alpha(G)=k$  elements. Define  $b_{ij} = 1/k$  if  $i, j \in Y$ , and  $b_{ij} = 0$ , otherwise. Then  $B=(b_{ij})$  is nonnegative and positive semi-definite with trace 1, and  $b_{ij}=0$  if  $\{i,j\} \in E$ . Furthermore,  $\sum_{i,j} b_{ij} = k$ . Hence  $\alpha(G) = k \leq \theta'(G)$ .  $\square$

THEOREM 2.  $\theta'(G) = \min\{\text{lev} A \mid A=(a_{ij}) \text{ is a symmetric } n \times n\text{-matrix with } a_{ij} \geq 1 \text{ if } \{i,j\} \notin E\}$ .

PROOF. By definition

$$(24) \quad \theta'(G) = \max\{B*J \mid B=(b_{ij}) \text{ is a symmetric positive semi-definite } n \times n\text{-matrix such that: } B*I=1, B*F_{ij}=0 \text{ for } \{i,j\} \in E, \text{ and } B*F_{ij} \geq 0 \text{ for } \{i,j\} \notin E\},$$

where  $F_{ij}$  is the  $n \times n$ -(0,1)-matrix with only ones in the positions  $(i,j)$  and  $(j,i)$ . From the above-mentioned form of the linear programming duality theorem it follows that this maximum equals

$$(25) \quad \min\{\lambda \in \mathbb{R} \mid M=(m_{ij}) \text{ is a symmetric } n \times n\text{-matrix; } m_{ij} \leq 0 \text{ if } \{i,j\} \notin E; \\ \lambda I + M - J \text{ is positive semi-definite}\}.$$

Putting  $A = J - M$ , one has, since, for symmetric  $A$ , the largest eigenvalue of  $A$  is equal to the minimum value of  $\lambda$  such that  $\lambda I - A$  is positive semi-definite,

$$(26) \quad \theta'(G) = \min\{\text{lev} A \mid A=(a_{ij}) \text{ is a symmetric } n \times n\text{-matrix such that} \\ a_{ij} \geq 1 \text{ if } \{i,j\} \notin E\}. \quad \square$$

Since the largest eigenvalue of a matrix is not increased by decreasing diagonal elements we may suppose that the minimum is attained by some  $A$  with ones on the diagonal.

We secondly prove that for graphs derived from symmetric association schemes  $\theta'(G)$  coincides with Delsarte's linear programming bound.

Let  $(X, \mathcal{R})$  be a symmetric association scheme, with  $\mathcal{R} = (R_0, \dots, R_n)$ , and let  $0 \in M \subset \{0, \dots, n\}$ . Let  $G=(X, E)$  be the graph with  $E = \bigcup_{i \notin M} R_i$ . Clearly,  $M$ -cliques in the association scheme coincide with independent sets in  $G$ .

**THEOREM 3.**  $\theta'(G)$  is equal to the linear programming bound for  $M$ -cliques in  $(X, \mathcal{R})$ .

**PROOF.** The linear programming bound is, by definition (cf. (12))

$$(27) \quad \max\{\sum_{k=0}^n a_k \mid a_0, \dots, a_n \geq 0; a_0=1; a_k=0 \text{ for } k \notin M; \sum_{k=0}^n a_k Q_k^u \geq 0 \text{ for} \\ u=0, \dots, n\}.$$

Let  $a_0, \dots, a_n$  attain this maximum, and put

$$(28) \quad (b_{ij}) = B = \sum_{k=0}^n \frac{a_k}{m \cdot v_k} D_k,$$

where  $m$ ,  $v_k$  and  $D_k$  are as in section 1. Then  $B$  satisfies the conditions mentioned in (23);  $B$  is positive semi-definite since, by the commutativity of  $D_0, \dots, D_n$ , the matrix  $B$  has eigenvalues

$$(29) \quad \sum_{k=0}^n \frac{a_k}{m \cdot v_k} p_k^u = \sum_{k=0}^n \frac{a_k}{m \cdot \mu_u} Q_k^u,$$

for  $u=0, \dots, n$ . Since  $D_k * J = v_k \cdot m$ , it follows that  $B * J = \sum_{i,j} b_{ij} = \sum_k a_k$ . Therefore, the linear programming bound is at most  $\theta'(G)$ .

To prove the converse, let  $b_0, \dots, b_n$  attain the minimum in (12), and let  $\lambda = \sum_u b_u$ . Define

$$(30) \quad A = \lambda I - \sum_{k,u=0}^n \frac{b_u}{\mu_u} Q_k^u \cdot D_k + J = \lambda I - \sum_{k=0}^n \left( \sum_{u=0}^n \frac{b_u}{\mu_u} Q_k^u - 1 \right) \cdot D_k.$$

Since  $\lambda I - A$  has eigenvalues

$$(31) \quad \sum_{k=0}^n \left( \sum_{u=0}^n \frac{b_u}{\mu_u} Q_k^u - 1 \right) \cdot p_k^v = \sum_{u=0}^n \frac{b_u}{\mu_u} \cdot m \cdot \delta_{uv} - \delta_{k0} \geq 0$$

( $v=0, \dots, n$ ) the matrix  $A$  has largest eigenvalue at most  $\lambda$ . Furthermore, by (4) and (12),  $a_{ij} \geq 1$  if  $\{i, j\} \notin E$ . Therefore, the minimum in (12) is at least the minimum of theorem 2, or the linear programming bound is at least  $\theta'(G)$ .  $\square$

If  $(X, R)$  is a Johnson scheme with  $n$  classes (cf. Delsarte [2]) and  $M = \{0, \dots, n-1\}$ , then  $G = K(v, n)$ . As Lovász showed that  $\theta(K(v, n)) = \binom{v-1}{n-1}$ , also Delsarte's linear programming bound yields the Erdős-Ko-Rado theorem.

Using techniques similar to those used in the proof of theorem 3 one proves for symmetric association schemes  $(X, \mathcal{R})$  and graphs  $G$  related as mentioned before theorem 3:

THEOREM 4.  $\theta(G) = \max\{\sum_{k=0}^n a_k \mid a_0=1; a_k=0 \text{ for } k \notin M; \sum_{k=0}^n a_k Q_k^u \geq 0 \text{ for } u=0, \dots, n\} =$   
 $\min\{\sum_{u=0}^n b_u \mid b_0, \dots, b_n \geq 0; b_0=1; \sum_{u=0}^n b_u P_k^u = 0 \text{ for } k \in M \setminus \{0\}\}.$

PROOF. Similar to the proof of theorem 3.  $\square$

So for graphs derived from symmetric association schemes there is an easier way to calculate the  $\theta$ -value. As a generalization of Delsarte's result (13) one has

THEOREM 5. *Let the edge set  $E$  of the graph  $G = (V, E)$  be the union of some classes of a symmetric association scheme. Then  $\theta(G) \cdot \theta(\overline{G}) = |X|$ .*

PROOF. Lovász proved that for all graphs  $G: \theta(G) \cdot \theta(\overline{G}) \geq |X|$ .

Now suppose  $E$  is the union of some classes of an association scheme, as described before theorem 3. Then by theorem 4,  $\theta(G) = \sum_k a_k$ , for some  $a_0, \dots, a_n$ , where  $a_0=1$ ,  $a_k=0$  for  $k \notin M$  and  $\sum_k a_k Q_k^u \geq 0$  for  $u=0, \dots, n$ . Set

$$(32) \quad b_u = \frac{\sum_{k=0}^n a_k Q_k^u}{\theta(G)}.$$

Then  $b_0, \dots, b_n \geq 0$  and  $b_0=1$ ; furthermore, for  $k \notin M$  (cf. (5)):

$$(33) \quad \sum_{u=0}^n b_u P_k^u = \frac{1}{\theta(G)} \cdot \sum_{u, \ell} a_\ell Q_\ell^u P_k^u = \frac{1}{\theta(G)} \cdot \sum_{\ell} a_\ell \cdot m \cdot \delta_{k\ell} = \frac{a_k \cdot m}{\theta(G)} = 0,$$

so  $b_0, \dots, b_n$  satisfy the conditions mentioned in the minimum-side of theorem 4, with  $\bar{G}$  instead of  $G$ . Also

$$(34) \quad \sum_{u=0}^n b_u = \frac{1}{\theta(G)} \cdot \sum_{k,u} a_k Q_k^u = \frac{1}{\theta(G)} \cdot \sum_k a_k \cdot \sum_u Q_k^u = \frac{1}{\theta(G)} \cdot \sum_k a_k \cdot m \cdot \delta_{k0} = \frac{|X|}{\theta(G)}.$$

Since, by theorem 4,  $\sum_u b_u \geq \theta(\bar{G})$  we have shown that  $\theta(G) \cdot \theta(\bar{G}) \leq |X|$ .  $\square$

Because there are (many) strongly regular graphs that are not vertex-transitive (cf. Seidel [8]) theorem 5 is not included in (17).

M.R. Best found the following example of a graph  $G$  with  $\theta'(G) < \theta(G)$ . The points of  $G$  are all vectors in  $\{0,1\}^6$ , two vectors being adjacent iff their Hamming distance is at most 3 (so the edge set is the union of some classes of a Hamming scheme). Then  $\theta'(G) = 4$  whereas  $\theta(G) = \frac{16}{3}$ .

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After completing this research I learnt that partially similar results have been obtained, independently, by McEliece, Rodemich & Rumsey [7] (cf. [6]). Their functions  $\alpha_L(G)$  and  $\theta_L(G)$  are equal to  $\theta'(G)$  and  $\theta(G)$ , respectively.

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